30

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Numericals Methods (2 marks)

Topics :-

① Solutions to Algebric and Transcandantal Eqn
- Bisection method
- ***** - Newton-Raphson Method
- Regula-false method
- Secant method

② Solutions ~~to~~ Systems of Linear Equation
- * - Gauss Elimination Method
- * - LU Decomposition

③ Solution to Integration of function
- ** - Trapezoidal Rule
- Simpson $\frac{1}{3}$ Rule
- Simpson $\frac{3}{8}$th Rule

④ Solution to differential Equations
- * - Euler's method
  - forward
  - Backward
- Runge Kutta method

1987 - 2012

36 Question

26 Question (Newton-Raphson method)

4 Question (Simpson Rule)

2 Que LU Decom

2 Ques Gauss

2 Ques Euler

(I) Mathematical methods are of two types.

(1) Analytical method

(2) Numerical method

(1) Analytical method :-

Eg. 1 ~~Ex. 1.~~ find roots of $x^2-5x+6=0$ using analytical method

→ Analytical solⁿ is $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\therefore\ x = 3, 2$

Eg. (2) find $\int_1^2 x\,dx$ using analytical method

→ $\int_1^2 x\,dx = \left[\dfrac{x^2}{2}\right]_1^2$

$= \left[\dfrac{4}{2} - \dfrac{1}{2}\right]$

$= \dfrac{3}{2}$

Eg. (3) Solve $\dfrac{dy}{dx} = x$, Using analytical method.

→ $\dfrac{dy}{dx} = x \qquad \therefore\ dy = x\,dx$

Integrate $\cancel{\int \frac{dy}{dx}\,dx}$ $\int dy = \int x\,dx$

$y = \dfrac{x^2}{2} + C$

Note:- - Drawback of analytical method is, it is not applicable for higher degree equations & also not applicable for non-linear Eq's.

To overcome this, we use NUMERICAL METHODS

- NUMERICAL METHODS provide Approximation value

[I] Solution to Algebric and Transcandantal Equation

* Intermediate Mean Value theorem :-

f(a) is a continuous function defined on [a, b] f(a) & f(b) having opposite signs. In such case there exist at least one Root of f(x) in [a, b]

Let,

$f(x) = x^3 - 4x - 9$, in [2, 3]

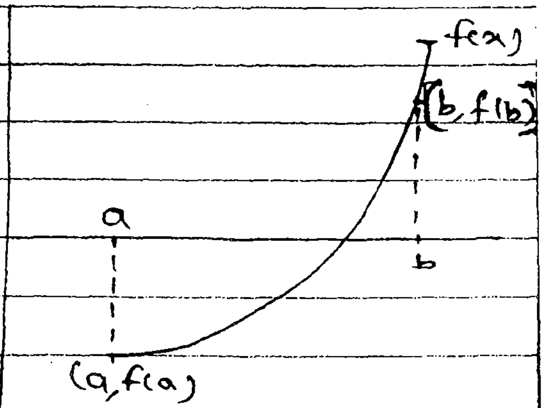
$\therefore f(2) = 2^3 - 4(2) - 9$

$= -9$

$\therefore -9 < 0$

& $f(3) = 3^3 - 4(3) - 9$

$= 6 > 0$



Since, $f(2) < 0$ & $f(3) > 0$

So, there exist atleast one root in [2, 3]

$\exists$ - there exists

* Bisection Method :-

Step I :- Let, $f(x)$ is a continuous on $[a, b]$

Step 2 :- $f(a)$ & $f(b)$ having opposite signs

Say $f(a) < 0$ & $f(b) > 0$

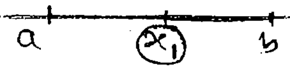
Using intermediate mean value theorem

there exist ($\exists$) atleast one root in $[a, b]$

Step 3 :- Let,

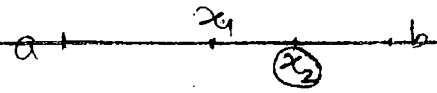
Approximation root is x_1 and

$$x_1 = \frac{a+b}{2}$$



Case I :-

If $f(x_1) = 0 \implies$ 'x_1' is root then Stop process.



Case II :-

If $f(x_1) < 0$ and $f(b) > 0$

then $x_2 = \frac{x_1 + b}{2}$

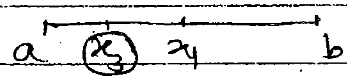
Continue this process until desired root is found

Case III :-

If $f(x_1) > 0$ and $f(a) < 0$

So $\exists$ atleast one root between $[a, x_1]$

Say, $x_3 = \frac{a + x_1}{2}$



Continue this process until desired root is found

Q find x_2 and x_3 using Bisection method where
$f(x) = x^3 - 4x - 9$, $[2, 3]$

→ Put intervals in $f(x)$

$f(2) = 2^3 - 4 \times 2 - 9$
$= -9$
$\therefore -9 < 0 \qquad \therefore f(2) < 0$

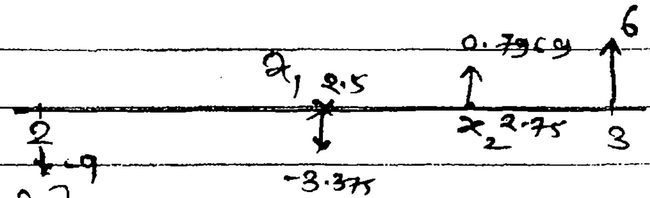
$f(3) = 3^3 - 4 \times 3 - 9$
$= 6$
$\therefore 6 > 0 \qquad \therefore f(3) > 0.$

$\exists$ atleast one root between $[2, 3]$

$x_1 = \frac{2+3}{2} = 2.5$

$f(2.5) = (2.5)^3 - 4(2.5) - 9$
$= -3.375$
$\therefore -3.375 < 0 \qquad \therefore f(2.5) < 0.$

Since $f(2.5) < 0$ and $f(3) > 0$



$\exists$ root b/w $[2.5, 3]$

$\therefore$ Let $x_2 = \frac{2.5+3}{2} = \frac{5.5}{2} = 2.75$

$f(x_2) = f(2.75) = (2.75)^3 - 4(2.75) - 9 = 0.7969$
$\therefore 0.7969 > 0$

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Since $f(2.75) > 0$ & $f(2.5) < 0$

So $\exists$ atleast one root in $[2.5, 2.75]$

Say, $x_3 = \frac{2.5 + 2.75}{2}$

$$\boxed{x_3 = 2.62}$$

* Newton Raphson Method :-

Step 1 :- Let, $f(x)$ is continuous function $[a, b]$

Step 2 :- Newton Raphson iteration formula for finding root of $f(x) = 0$ is,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Q. Find Newton raphson iteration formula for square root of +ve real number 'C'.

IMP

→ Let, $x = \sqrt{C}$

Squaring b-s.

$x^2 = C$

$\therefore x^2 - C = 0$

$\therefore f(x) = x^2 - C \qquad \therefore f(x_n) = x_n^2 - C$

$f'(x) = 2x \qquad f'(x_n) = 2x_n$

$\therefore$ By Newton Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{x_n^2 - c}{2x_n}$$

$$\boxed{x_{n+1} = \frac{x_n^2 + c}{2x_n}}$$

Q. find N-R iteration formula for $f(x) = x^2 - 117 = 0$

GATE 2009 →

$f(x) = x^2 - 117$; $f(x_n) = (x_n)^2 - 117$

$f'(x) = 2x$; $f'(x_n) = 2x_n$

$$x_{n+1} = x_n - \frac{x_n^2 - 117}{2x_n}$$

$$\boxed{x_{n+1} = \frac{x_n^2 + 117}{2x_n}}$$

Q. If $f(x) = x^2 - 13$ and $x_0 = 3.5$, then Value of x_1 using N-R. iteration formula

GATE 2010 →

$f(x) = x^2 - 13$ $\quad$ $f(x_n) = x_n^2 - 13$

$f'(x) = 2x$ $\quad$ $f'(x_n) = 2x_n$

$$\therefore x_{0+1} = x_0 - \frac{x_0^2 - 13}{2x_0}$$

$$x_1 = \frac{x_0^2 + 13}{2x_0}$$

$$x_1 = \frac{(3.5)^2 + 13}{2 \times 3.5} = \underline{\underline{3.6071}}$$

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Q. Newton Raphson iteration formula for finding $\sqrt[3]{c}$

GATE 1996

→ Let $x = \sqrt[3]{c}$

Cubing

$x^3 = c$

$x^3 - c = 0$

$f(x) = x^3 - c$

$f(x_n) = x_n^3 - c$

$f'(x) = 3x^2 - 0$

$f'(x_n) = 3x_n^2$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - c}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 + c}{3x_n^2}$$

$$\boxed{x_{n+1} = \frac{2x_n^3 + c}{3x_n^2}}$$

Q. The N-R method is used to find the root of $x^2 - 2 = 0$ & starting value is $x_0 = -1$; The iteration formula will be

GATE 1997

Fastness or Convergence is Rate of convergence

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(a) Converges to -1 (b) Converges to $-\sqrt{2}$
(c) Converges to $\sqrt{2}$ (d) Not convergent

$f(x) = x^2 - 2$ $f(x_n) = x_n^2 - 2$

$f(x_n) = x_n^2 - 2$ $f'(x_n) = 2x_n$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = -1 - \frac{(-1)^2 - 2}{2(-1)}$$

$$= -1 - \frac{1-2}{-2}$$

$$= -1 - \frac{+1}{2}$$

$$= -1 - 0.5$$

$$x_1 = -1.5$$

$$\therefore x_2 = \frac{x_1^2 + 2}{2x_1} = \frac{(-1.5)^2 + 2}{2(-1.5)} = -1.416$$

$$\therefore x_3 = \frac{x_2^2 + 2}{2x_2} = \frac{(-1.416)^2 + 2}{2(-1.416)} = -1.414$$

||ly $x_4 = -1.414$ i.e. $-\sqrt{2}$

Recess 10:30 am

ntinue
1:00pm

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Q find Newton Raphson iteration formula, ~~for~~ reciprocal of a where $a > 0$

GATE 2005

→ Let, $x = \frac{1}{a}$

$$\frac{1}{x} = a$$

$$\therefore \frac{1}{x} - a = 0$$

$$\therefore f(x) = 1/x - a$$

$$f'(x) = \frac{-1}{x^2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(1/x_n - a)}{(-1/x_n^2)}$$

$$= x_n + x_n^2 \left(1/x_n - a\right)$$

$$= x_n + x_n - a x_n^2$$

$$\boxed{x_{n+1} = 2x_n - a x_n^2}$$

Q Given $a > 0$, we wish to compute N-R iteration formula for reciprocal of a for $a = 7$ and $x_0 = 0.2$ then first two iteration will be.

GATE 2005

(a) 0.11, 0.1299 ~~(b)~~ 0.12, 0.1392

$0.4 - 7 \times 0.04$ अश्लील, गंदे विचारवाली पुस्तक पढ़ना जहर पीने से भी ज्यादा खतरनाक है।

$= 0.4 - 28 = \frac{.28}{0.12}$

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Given $x = \frac{1}{a}$

$\therefore \frac{1}{x} - a = f(x)$

$\therefore x_{n+1} = 2x_n - ax_n^2$

$x_{0+1} = 2x_0 - ax_0^2$

$= 2 \times 0.2 - 7(0.2)^2$

$x_1 = 0.12$

$x_{1+1} = x_2 = 2x_1 - ax_1^2$

$= 2(0.12) - 7(0.12)^2$

$= 0.24 - 7 \times 0.0144$

$x_2 = 0.1392$

H.W
395, Q $f(x) = x - \cos x$, then $x_{n+1} =$ ______ ?

$\Rightarrow$ Ans. $\Rightarrow x_n - \frac{(x_n - \cos x_n)}{1 + \sin x_n}$

H.W Q $f(x) = x.e^x - 2$, $x_0 = 0.8679$ then $x_1 =$?

$\Rightarrow$ Ans. $\Rightarrow x_1 = 0.853$

H.W Q $f(x) = x^3 - x^2 + 4x - 4 = 0$, $x_0 = 2$, then $x_1 =$?

$\Rightarrow$ Ans $\Rightarrow x_1 = 4/3$

H.W Q $f(x) = e^x - 1$, $x_0 = -1$, then $x_1 =$ ______ ?

$\Rightarrow$ Ans $\Rightarrow x_1 = 0.71828$

$$2x + y + z = 10$$
$$y + 3z = 6$$
$$-2z = -10$$

$\therefore$ By Using Back Substitution,
we get $z = 5,\ y = -9,\ x = 7$

Case II :-

If $\rho(A) = \rho(AB) = r$
but $r < n$

* * (i) No. of linearly independent solutions $= n - r$.

(ii) No. of linearly dependent Solutions $= r$

In this case system has infinitely many solutions

Ex:- Same example only 3rd row elements are made Zeros.

$$\therefore \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here,
$$\rho(A) = 2 = \rho(AB)$$

(i) No. of linearly independent soln $= n - r$
$= 3 - 2$
$= 1.$

(ii) No. of linearly dependent $= r$
$= 2.$

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$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 0 \end{bmatrix}$$

$2x + y + z = 10$ ——— (i)

$y + 3z = 6$ ——— (ii)

$\therefore y = 6 - 3z$ ——— (i) ——— y is dependent

z is independent

& from (i)

$x = 2 + z$ ——— (ii)

x is dependent

z is independent

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+z \\ 6-3z \\ z \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

for different values of z we get different sol^n. so system has infinite sol^n.

Bisection $f(x)$ $[a,b]$ $\frac{d}{dx}\sqrt{x} = \frac{1}{2}$

N·R $f(x)$ x_0.

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Q $f(x) = x - e^{-x}$, then $x_{n+1} =$?

2008 $\Rightarrow$ Ans. $\Rightarrow$ $x_{n+1} = \dfrac{e^{-x_n}(1+x_n)}{1+e^{-x_n}}$

2001 Q $f(x) = x + \sqrt{x-3}$ and $x_0 = 2$, then $x_1 =$ ____ ?

$\Rightarrow$ Ans $\Rightarrow$ $x_1 = 1.8124$

Q The newton raphson method used to find the root of the equation and $f'(x)$ is derivative of f then the method is converges

1999

ⓐ Always ⓑ Only if 'f' is polynomial

ⓒ Only if $f(x_0) < 0$

ⓓ None of the above $\Rightarrow$ Ans. = ⓓ.

$\longrightarrow$

- Newton Raphson method is useful for finding roots of Eq^n whether curve is ler to x axis, i.e the curves which are generating high slopes we can get better results using N-R method

- If slope is less then N-R method is not providing accurate results

- The N-R method converging to the root if it satisfy the following eq^n

$$|f(x) \cdot f''(x)| \leq |f'(x)|^2$$

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* Regula false Method :-

Step 1 :- Let, $f(x)$ is Continuous function in $[a, b]$

Step 2 :- Let us Assume that x_0 and x_1 are initial approximation values for the recquired root such that $f(x_0)$ and $f(x_1)$ having opposite signs

Say $f(x_0) < 0$, $f(x_1) > 0$

Step 3 :- Regula false Iteration formula for finding root of $f(x)$ in $[x_0, x_1]$ is

if $$\boxed{x_n = \frac{f_n \cdot x_{n-1} - f_{n-1} \cdot x_n}{f_n - f_{n-1}}}$$

In particular $$\boxed{x_2 = \frac{f_1 \cdot x_0 - f_0 x_1}{f_1 - f_0}} \quad n=1 \qquad \text{(i)}$$

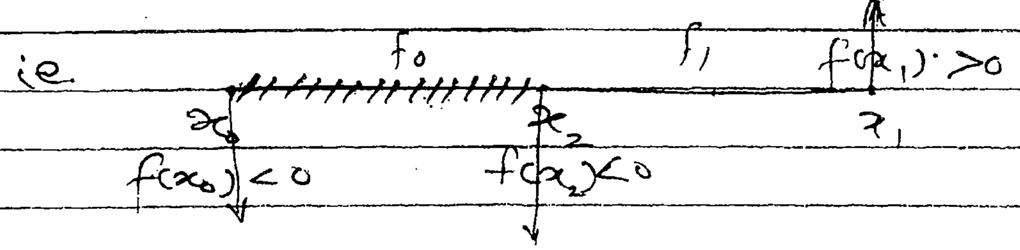
Case I :-

If $f(x_2) = 0 \implies x_2$ is root, then Stop process

Case II :-

If $f(x_2) < 0$ and $f(x_1) > 0$

ie



To Compute 'x_3', replace 'x_0' by x_2

$$\therefore x_3 = \frac{f_1 x_2 - f_2 x_1}{f_1 - f_2}$$

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Case III :-

If $f(x_2) > 0$ and $f(x_0) < 0$

So to compute x_3, we have to replace x_1 by x_2, f_1 by f_2 in Eq^n (i)

$$\therefore \boxed{x_3 = \frac{f_2 \cdot x_0 - f_0 x_2}{f_2 - f_0}}$$

Continue the process until desired accuracy is found

Q. $f(x) = x^3 + x - 1$ and $[0.5 \quad 1]$ then find x_2, x_3 using Regula false method

→ $[0.5, 1] = [x_0, x_1]$

$f(x) = x^3 + x - 1$

$f(0.5) = (0.5)^3 + 0.5 - 1 = -0.375$ i.e < 0

$f(1) = 1^3 + 1 - 1 = 1$ i.e. > 0

let, $x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$

$= \frac{1 \times 0.5 - (-0.375) \times 1}{1 - (-0.375)}$

$x_2 = 0.64$

Now, $f_2 = f(x_2) = f(0.64) = (0.64)^3 + 0.64 - 1$

$= -0.0979$

Re false $f(x)$
Secant $f(x)$

Assume $[x_0, x_1]$ $\to f(x_0)<0$, $f(x_1)>0$

$[x_0, x_1]$, $f(x_0) \cdot f(x_1) < 0$

$$\therefore x_3 = \frac{f_1 \cdot x_2 - f_2 \cdot x_1}{f_1 - f_2}$$

formula
$$x_3 = \frac{f_1 x_2 - f_2 x_1}{f_1 - f_2}$$

$$= \frac{-0.0979 \times 0.5 - (-0.375) \times (0.64)}{-0.0979 - (-0.375)}$$

wrong.

$$= \frac{-0.0979 \times 0.5 + 0.375 \times 0.64}{-0.979 + 0.375}$$

$$\boxed{Ans. = 0.672}$$

* Secant Method :-

The difference between Regula false & Secant method is, in Secant method the initial Guess values x_0, x_1 need not satisfy the condition.

Let, $[f(x_0) \times f(x_1)] < 0$

i.e. Secant method does not provide 100% Gaurette that the root is existing in the initial guess interval (x_0, x_1)

Iteration formula for finding roots of Given Eqn Using Secant method is

$$\boxed{x_{n+1} = \frac{f_n \cdot x_{n-1} - f_{n-1} \cdot x_n}{f_n - f_{n-1}}}$$

$x_0 \curvearrowright x_1 \curvearrowright x_2 \curvearrowright x_3 \curvearrowright x_4$

$$\begin{array}{ccc} f_0 & x_2 & f_1 \\ x_0 & & x_1 \end{array}$$

In particular,

$$\boxed{x_2 = \frac{f_1 \cdot x_0 - f_0 \cdot x_1}{f_1 - f_0}} \qquad \text{— (i)}$$

To compute x_3, x_0 is replaced by x_1
x_1 replaced by x_2 in eq^n (i)

Continue process until desired accuracy of root is found.

Q. Using Secant Method, find 1st & 2nd approximation of the real root for the equation $x^3 - 2x - 5 = 0$, with $[2, 3]$

→

$f(x) = x^3 - 2x - 5$, ~~f(2) =~~

$f(2) = 2^3 - 2(2) - 5 = -1 < 0$ — f_0

$f(3) = 3^3 - 2(3) - 5 = 16 > 0$ — f_1

$$x_2 = \frac{f_1 \cdot x_0 - f_0 x_1}{f_1 - f_0} = \frac{16 \times 2 - (-1) \times 3}{16 (-) (-1)}$$

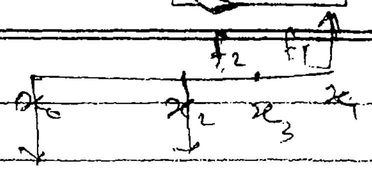
$$= \frac{32 + 3}{17}$$

$$= \frac{35}{17} = 2.058$$

$\therefore f(x_2) = (2.058)^3 - 2(2.058) - 5$

$= -0.3996 \approx -0.3997$

$$x_3 = \frac{f_2 \cdot x_1 - f_1 \cdot x_2}{f_2 - f_1}$$



$$= \frac{-0.3907 \times 3 - 16 \times 2.058}{-0.3907 - 16}$$

$$x_3 = 2.0812$$

* Method	Order of Convergence
① Bisection	linear Convergence $\Rightarrow \varepsilon_{n+1} = K \cdot \varepsilon_n^1$ means of order 1
② Regula falsi	linear Convergence $\Rightarrow \varepsilon_{n+1} = K \cdot \varepsilon_n^1$ –"– –"– 1
③ Secant Method	Quadratic Convergence $\Rightarrow \varepsilon_{n+1} = K \cdot \varepsilon_n^{1.62}$ –"– –"– 1.62
④ Newton Raphson	Quadratic Convergen. $\Rightarrow \varepsilon_{n+1} = K \cdot \varepsilon_n^2$ –"– –"– 2

$\alpha = 2$

if $x_n = 2.02$,

$x_{n+1} = 2.004$

Error = Exact − Approxi

$\varepsilon_n = 2 - 2.02$

$= -0.02$

$\varepsilon_{n+1} = -0.004$

$$\boxed{\varepsilon_{n+1} = \varepsilon_n^2}$$

Bisec 1
Reg 1.62
Sec
Newto 2

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magnitude

and

$$x_n = 2.03$$
$$x_{n+1} = 2.06$$

$$\varepsilon_{n+1} = 0.06$$
$$\varepsilon_n = 0.03$$

$$\boxed{\varepsilon_{n+1} = 2\varepsilon_n^1}$$

Note:- Let us consider n^{th} degree polynomial
$f(x) = 0$

(a) The number of +ve real roots for $f(x) \leq$ The number of sign changes in $f(x) = 0$

(b) The number of −ve real roots for $f(x) \leq$ The number of sign changes in $f(-x) = 0$

(c) The number of imaginary roots =
$n -$ (No. of +ve roots + No. of −ve roots)

Q Polynomial $P(x) = x^5 + x + 2$ has
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(a) All real roots (c) Three real roots & 2 complex
(b) 1 real & 4 complex roots (d) All complex roots

→ from Note (a)

no. of +ve real roots $\leq$ 0 No. of sign changes in $P(x)$

∴ No. of +ve real roots = 0

from Note (b) No. of −ve real roots $\leq$ No. of sign changes in $P(-x)$

$$P(-x) = -x^5 - x + 2$$

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$P(x) = 1$ Change

$\therefore$ '1' (-ve) real roots.

Imaginary roots $= n - [(+ve) + (-ve) \text{ roots}]$

$= 5 - [0+1]$

$= 4$

$\therefore$ Ans. 1 Real root & 4 complex root

Q. If $\alpha_1, \alpha_2, \alpha_3 \cdots\cdots \alpha_n$ are n roots of Eq^n.

$a_0 x^n + a_1 x^{n-1} + \cdots\cdots + a_{n-1} x + a_n = 0$

(a) $\sum_{i=1}^{n} \alpha_i =$ ____ (b) $\sum \alpha_1 \alpha_2 =$ ____

(c) $\sum \alpha_1 \alpha_2 \alpha_3 =$ ____ (d) $\alpha_1, \alpha_2, \cdots\cdots \alpha_n = (-1)^n \frac{a_n}{a_0}$.

$ax^2 + bx + c = 0$

$\therefore \alpha_1 + \alpha_2 = -\frac{b}{a}$ $\qquad \alpha_1 \cdot \alpha_2 = \frac{c}{a} = \frac{\text{Const.}}{\text{Coeff of } x^2}$

$\therefore \alpha_1 \cdot \alpha_2 \cdot \alpha_3 = \frac{\text{Const}}{\text{Coeff} \cdot \text{of } x^3}$

$\vdots$

$\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdots \alpha_n = \frac{\text{Const}}{\text{Coeff} \cdot x^n}$

$\therefore \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdots \alpha_n = \frac{a_n}{a_0}$

पृष्ठ सं. :
दि.: / /

$\therefore \; ax^3 + bx^2 + cx + d = 0.$

$$\therefore \alpha_1 + \alpha_2 + \alpha_3 = \frac{-b}{a} = \frac{-a_1}{a_0}$$

$$\sum \alpha_1 \cdot \alpha_2 = \frac{a_2}{a_0}$$

$$\therefore \sum \alpha_1 \cdot \alpha_2 \cdot \alpha_3 = \frac{a_3}{a_0}$$

$$\therefore \sum_{i=1}^{n} \alpha_i = \frac{-a_1}{a_0}$$

$$\sum \alpha_1 \cdot \alpha_2 = \frac{a_2}{a_0}$$

$$\sum \alpha_1 \cdot \alpha_2 \cdot \alpha_3 = -\frac{a_3}{a_0}$$

$$\alpha_1 \cdot \alpha_2 \cdots \alpha_n = (-1)^n \cdot \frac{a_n}{a_0}$$

alternate +ve, -ve Signs.

Q. GATE 2008 It is known that the roots of the non-linear eqn $x^3 - 6x^2 + 11x - 6 = 0$ are 1 & 3 then 3rd root will be

$$\alpha_1 + \alpha_2 + \alpha_3 = \frac{-b}{a} = \frac{-(-6)}{1} = 6$$

$$\alpha_1 \cdot \alpha_2 \cdot \alpha_3 = \frac{c}{a} = \frac{-6}{1} = -6$$

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Here, $x^3 - 6x^2 + 11x - 6 = 0$

$\therefore a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0$

$$\alpha_1 \cdot \alpha_2 \cdot \alpha_3 = \frac{c}{a} = (-1)^3 \frac{(-6)}{1}$$

$$1 \times 3 \times \alpha_3 = (-1)(-6)$$

$$\alpha_3 = \frac{6}{3}$$

$$\boxed{\alpha_3 = 2}$$

24/08/12

II Solutions to System of linear Equation

(4) Gauss Elimination :-

⟹ "MATRIX METHOD"

Q. Solve
$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

⟶

Step I :- Construct Augumented matrix

i.e
$$[A : B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

A = Coefficient matrix
B = Constant matrix.

Step II :- Convert augumented matrix into an upper triangular matrix using elementary row operations

Here, in above prob we have to do Row operation as, $R_2 - \frac{3R_1}{2}$ and $R_3 - \frac{R_1}{2}$

$$\therefore [A : B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/2 & 11 \end{array}\right]$$

$$R_3 - \frac{(7/2) R_2}{1/2} = R_3 - 7R_2$$

$$\therefore \approx \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

Imp
****
Expected
Ques.

$\rho(A)$ = No. of Non Zero rows in an uppertriangular matrix OF A.

Case I :-

IF $\rho(A) = \rho(AB) = r = n$

where, n is no. of unknowns
i.e. $(x, y, z \ldots$ etc$)$

Then the System is said to CONSISTENT and it has a UNIQUE SOLUTION.

$\Rightarrow$ Continue to prob.

$$\therefore \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

Here $\rho(A) = \rho(AB) = 3 = n$

So, it has a Unique sol^n.

$$\therefore \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ -10 \end{bmatrix}$$

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Case (iii) :-

If $\rho(A) \neq \rho(AB)$ then System is said to be "INCONSISTENT", then it has "NO SOLUTION"

Ex:- Same example, but in constant matrix in third row a const is present i.e -5.

$$\therefore \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

$$\therefore \rho(A) = 2$$
$$\rho(AB) = 3$$

$\therefore \rho(A) \neq \rho(AB)$ —— "NO SOLUTION"

Gauss Elimination :-

$\Longrightarrow$ "PIVOTAL SOLUTION"

Q. Solve :-

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

Step I :- Construct augmented matrix

i.e. $[A:B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$

from (2, 3, -10, 5, 1, -7) the largest absolute value is (-10)

absolute = |modulus|

Since, $a_{11} = 2$

Now, Scan entire 1st column and select largest absolute value and make it as "Pivot". Exchange pivot element row with 1st row and then eliminate x from row 2 & row 3.

$R_2 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 18 \\ 2 & 1 & 1 & 10 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$R_2 - \frac{2R_1}{3}$ & $R_3 - \frac{R_1}{3}$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 18 \\ 0 & -1/3 & -1 & -2 \\ 0 & 10/3 & 8 & 10 \end{array}\right]$$

Since, $a_{22} = -1/3$

Now, Scan entire 2nd column from a_{22} and select largest absolute value i.e. $\left|\frac{-1}{3}\right| < \left|\frac{10}{3}\right|$ and make it as pivot

Exchange pivot element row with 2nd row and then eliminate y from row 3

a_{11}

a_{22}

a_{33}

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$R_2 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 3 & 2 & 3 & 18 \\ 0 & 10/3 & 8 & 10 \\ 0 & -1/3 & -1 & -2 \end{array}\right]$

$$R_3 + \frac{\left(\frac{1}{3}\right)R_2}{\left(\frac{10}{3}\right)} = R_3 + \frac{R_2}{10}$$

$$\approx \left[\begin{array}{ccc|c} 3 & 2 & 3 & 1 \\ 0 & 10/3 & 8 & 10 \\ 0 & 0 & -1/5 & -1 \end{array}\right]$$

$$\begin{bmatrix} 3 & 2 & 3 \\ 0 & 10/3 & 8 \\ 0 & 0 & -1/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 10 \\ -1 \end{bmatrix}.$$

By solving, $z = 5$
$y = -9$
$x = 7$

GATE 2009 Q. In the solutions of the following set of linear equations by Gauss Elimination Using Pivotal solⁿ the pivots for eliminating "x" & "y" respt.

$5x + y + 2z = 34$
$4y - 3z = 12$
$10x - 2y + z = -4$

(a) 10 & 4
(b) 10 & 2
(c) 5 & 4
(d) 5 & -4

$$\left[\begin{array}{ccc|c} 5 & 1 & 2 & 34 \\ 0 & 4 & -3 & 12 \\ 10 & -2 & 1 & -4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 10 & -2 & 1 & -4 \\ 0 & 4 & -3 & 12 \\ 5 & 1 & 2 & 34 \end{array}\right]$$

~~34~~ 68 +4

$R_3 - \cancel{R_1}/2$

$$\left[\begin{array}{ccc|c} 10 & -2 & 1 & -4 \\ 0 & 4 & -3 & 12 \\ 0 & \cancel{2} & 3/2 & 36 \end{array}\right]$$

~~Ans.~~ $a_{11} = 10$
$a_{22} = 4$

$\therefore$ Ans. 10 & 4.

(B) LU Decomposition (Method of factorisation) or Do-little method.

Step I :- Let us consider
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Step II :- Matrix representation of given system of equation is
$$AX = B \quad \text{——— (i)}$$

Let, $A = LU$ ———— (ii)

———— where L = lower Unit Δ'ular matrix

$$\text{ie } L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

U = Upper Δular matrix

$$= \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$\therefore$ (i) $\Rightarrow$ $LU\cancel{X} = B$ ———— (iii)

Let, $UX = Y$ ———— (iv)

where. $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\therefore$ $LY = B$.

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

By solving, we get, y_1, y_2, y_3 in terms of elements of lower Unit Δ'ular matrix.

$$\therefore \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

By Solving, we get x_1, x_2, x_3 in terms of the elements of Upper Δ'ular and lower Unit Δular matrix.

The order computing elements of L & U is
$U_{11}, U_{12}, U_{13}, l_{21}, U_{22}, U_{23}, l_{31}, l_{32}, U_{33}$.

Note:- CROUT'S method is similar to Do-little method except that in Crout's method 'A' is decomposed with lower triangular matrix & Unit upper triangular matrix.

i.e. $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

$U = \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$

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GATE 2011 Q In matrix A is $\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$ is decomposed into product of Lower and upper triangular matrices Using Crout's method The properly decomposed L & U matrices respectively

(a) $\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & U_{12} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11} U_{12} \\ l_{21} & l_{21} \cdot U_{12} + l_{22} \end{bmatrix}$$

$\therefore \ l_{11} = 2$

$l_{11} U_{12} = 1 \qquad \therefore U_{12} = 1/2$

$l_{21} = 4$

$l_{21} \cdot U_{12} + l_{22} = -1 \qquad 4 \times \frac{1}{2} + l_{22} = -1$

$l_{22} = -3$

$$\therefore \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & U_{12} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

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III Solution to Integration of function :-

Let us consider the given curve is f(x) and ordinates on x axis is $x = a$ & $x = b$

The area bounded by the given curve and the ordinates is denoted by

$$\int_a^b f(x)\,dx \quad \text{———} \quad (*)$$

Divide $[a, b]$ into "n" equal subintervals where, length of each interval is "h" (Step Size)

$a = x_0$

$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + h + h = x_0 + 2h$

$x_3 = x_2 + h = x_0 + 2h + h = x_0 + 3h$

⋮

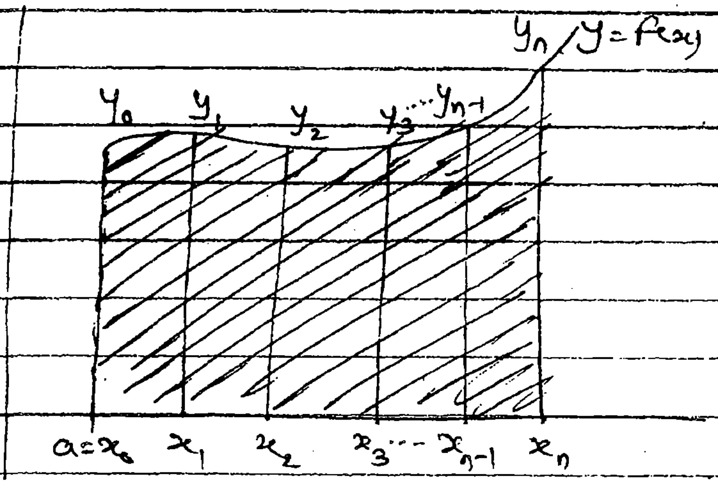
$x_n = x_0 + nh$

i.e. $b = x_0 + nh$

$\therefore x_n = x_0 + nh$

$b = a + nh$

$\therefore \boxed{n = \dfrac{b-a}{h}}$



In Q., Only Simpson rule is mentioned then take Simpson $\frac{1}{3}^{rd}$ rule

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Equation (*) Can be evaluated by Using

1) Trapezoidal Rule

$$= \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots\cdots y_{n-1})\right]$$

2) Simpson $\frac{1}{3}$ Rule

$$= \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+y_6+\cdots\cdots y_{n-2}) + 4(y_1+y_3+y_5+\cdots\cdots y_{n-1})\right]$$

3) Simpson $\frac{3}{8}$th Rule

$$= \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+y_9+\cdots\cdots y_{n-3}) + 3(y_1+y_2+y_4+y_5+y_7+y_8+y_{10}+y_{11}\cdots y_{n-1})\right]$$

Recess. 10:40 am

Continue 11:15 am.

Gate Q 2010

h

x	0 (x_0)	0.25 (x_1)	0.5	0.75	1.0
$f(x)$	1	0.9412	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

The value of the integrated betⁿ the limits 0 & 1 Using Simpson Rule (if not mentioned then take $\frac{1}{3}^{rd}$ rule)

$$\text{Simpson } \frac{1}{3} \text{ rule} = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4\cdots)+4(y_1+y_3\cdots)\right]$$

$$= \frac{0.25}{3}\left[(y_0+y_4)+2(y_2)+4(y_1+y_3)\right]$$

$$= \frac{0.25}{3}\left[(1+0.5)+2(0.8)+4(0.9412+0.64)\right]$$

$$= 0.7854$$

Note: ① Simpson's Rule is applicable if the number of intervals are "EVEN"

② Simpson's $\frac{3}{8}$th Rule is applicable if the number of intervals are multiples of 3 i-e ($n = 3, 6, 9, 12$, etc)

③ Trapezoidal Rule is applicable for any number of intervals

Q. A 2nd degree polynomial f(x) takes a following values

GATE 2006

x	0	1	2
f(x)	1	4	15

~~The integration 0 to 2~~ $\int_0^2 f(x)\,dx$ is Evaluated using trapezoidal Rule then the error estimation is

ⓐ $4/3$ ⓑ $-4/3$ ⓒ $2/3$ ⓓ $-2/3$

Error = Exact value − Approximate value

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→ Here,

It is mentioned that 2^{nd} degree polynomial.

$\therefore f(x) = a_0 + a_1 x + a_2 x^2$.

and it takes a followings values —— given.

$\therefore f(0) = 1$ —— i.e $x=0$, $f(x)=1$.

$\therefore a_0 = 1$

$\therefore f(1) = 4$

$\therefore a_0 + a_1 + a_2 \cancel{1}^2 = 4$

$1 + a_1 + a_2 = 4$

$a_1 + a_2 = 3$. —— (i)

$f(2) = 15$

$a_0 + a_1 \times 2 + a_2 \times 2^2 = 15$

$2a_1 + 2a_2 = 14$ (with a_2 marked 2 below)

$a_1 + 2a_2 = 7$ —— (ii)

$\therefore$ Solve (i) & (ii)

$\cancel{a_0} \cdot a_2 = 4$

$a_1 = -1$

$\therefore a_0 = 1,\ a_1 = -1,\ a_2 = 4$

$\therefore f(x) = 1 - x + 4x^2$.

$\therefore$ Exact value $= \int_0^2 f(x)\,dx = \int_0^2 (1 - x + 4x^2)\,dx$

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$$= \left[x - \frac{x^2}{2} + \frac{4x^3}{3} \right]_0^2$$

Exact $= \frac{32}{3}$

Approximate value = Trapezoidal Rule value

$\therefore$ T. R. value $= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 \cdots) \right]$

$$= \frac{1}{2}\left[(1+15) + 2(4) \right]$$

$$= 12$$

$\therefore$ Error = Exact − Approximate

$$= \frac{32}{3} - 12$$

$$= \frac{-4}{3}.$$

$\frac{2\pi}{8} \cdot \frac{\pi}{4}$ 0 $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, 2\pi$

5 significant digit = upto 5th decimal pt.

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ME Q. GATE 2007

$\int_0^{2\pi} \sin x \, dx$ is Evaluated by T.R. Rule with Eight Equal intervals, with **5** significant digits

(a) 0.00000 (b) 1.00000 (c) 0.00500 (d) 0.00025

→ Here, $n=8$, $h = \frac{b-a}{n} = \frac{2\pi - 0}{8} = \frac{\pi}{4}$.

$h = \frac{\pi}{4}$

x	x	$\sin x$	
~~Sinx~~	$x=0$	$\sin 0 = 0$	y_0
	$x+h = 0+\pi/4$	0.70710	y_1
	$\pi/2$	1	y_2
	$3\pi/4$	0.70710	y_3
	π	0	y_4
	$5\pi/4$	-0.70710	y_5
	$6\pi/4$	-1	y_6
	$7\pi/4$	-0.70710	y_7
	2π	0	y_8

$$\text{T.R. Rule} = \frac{h}{2}\left[(y_8 + y_0) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \frac{\pi}{4} \cdot \frac{1}{2}\left[(0+0) + 2(0.70710 + 1 + 0.70710 + 0 - 0.70710 - 1 - 0.70710)\right]$$

$= 0.00000$

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Q

ME
GATE
2010

x	0	60	120	180	240	300	360
y	0	1068	-323	0	323	-355	0

Evaluate $\int_0^{2\pi} y\,dx$ Using Simpson's Rule

(a) 542 (b) 995 (c) 1444 (d) 1986

$\rightarrow = \frac{h}{3}\left[(y_0+y_6)+2(y_3+\cancel{y})+4(y_1+y_2+y_4+y_5+\cancel{y})\right]$

$\cancel{h = \frac{b-a}{\pi} = 2\pi - 0}$

$= \cancel{\frac{60}{3}}\left[(0+\right.$

$\ominus$

Here $h = 60.$

But the Integration limit is in π term

$\therefore$ ~~take~~ $h = \frac{\pi}{3}$

$\frac{1}{3}^{rd}$ $= \frac{\pi}{3}\left[(y_0+y_6)+2(y_3)+4(y_1+y_2+y_4+y_5)\right]$

$= 995.$

x	ϕ	2	3
$\frac{1}{x}$	$\frac{1}{0}$	$\frac{1}{2}$	$\frac{1}{3}$

MEQ 2011 GATE

The integral $\int_1^3 \frac{1}{x}\,dx$ when Evaluated Using $\frac{1}{3}$rd rule on two equal intervals Each of length 1.

(a) 1.000 (b) 1.111

→ $\int_1^3 \frac{1}{x}\,dx$. $h = \frac{b-a}{n}$ —— $n = 2$

$= \frac{3-1}{2}$

$h = 1$

	x	$1/x$
y_0	1	$1/1 = 1$
y_1	2	$1/2$
y_2	3	$1/3$

$\frac{1}{3}\text{rd} = \frac{h}{3}\left[(y_0 + y_2) + \cancel{\;}\right]$

$\cancel{= \frac{1}{3}[0 + 0.5]}$

$\cancel{= \frac{0.5}{3}}$

$\cancel{= 0.}$

$\frac{1}{3}\text{rd} = \frac{1}{3}\left[\left(1 + \frac{1}{3}\right) + 2(0) + 4(y_1)\right]$

$= \frac{1}{3}\left[\frac{4}{3} + 4 \times \frac{1}{2}\right]$

$= \frac{1}{3}\left[\frac{4}{3} + 2\right]$

$= \frac{1}{3}\left[\frac{10}{3}\right]$

$= \frac{10}{9}$

$= 1.111$

Error order T.R rule = h^2
Error order Simpson $\frac{1}{3}$ = h^4
Error order Simpson $\frac{3}{8}$ = h^5.

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Imp Q

Q GATE 2008 The minimum number of equal length of subinterval needed to approximate

$$\int_1^2 x \cdot e^x \, dx$$ to an accuracy of at least $\frac{1}{3} \times 10^{-6}$ Using T.R rule

(a) $1000 \cdot e$ (b) 1000 (c) $100 \cdot e$ (d) 100

→ $n = ?$

$f(x) = x \cdot e^x$

Note :- (a) $\boxed{\text{Error in T.R. Rule} = -\left(\frac{b-a}{12}\right) \cdot h^2 \cdot \max\left[f''(x)\right]}$

(b) $\boxed{\text{Truncation Error in T.R Rule} = \frac{(b-a)}{12} \cdot h^2 \cdot \max\left[f''(x)\right]}$

— Error order h^2

At least means $\geq$

Here, Error ↑, Accuracy ↓

Given, Accuracy $\geq \frac{1}{3} \times 10^{-6}$

Truncation Error $\leq \frac{1}{3} \times 10^{-6}$

$$\frac{(b-a)}{12} \cdot h^2 \cdot \max\left[f''(x)\right] \leq \frac{1}{3} \times 10^{-6}$$

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Here,

$f(x) = xe^x$

$f'(x) = xe^x + e^x$

$f''(x) = xe^x + e^x + e^x$

$\therefore f''(x) = 2e^x + xe^x$

— Here, in $f''(x)$, e^x is Increasing function as x increases.

& xe^x is also.

$\therefore f''(x)$ is Increasing function in $(1,2)$

$\therefore f''(2) = 2\times e^2 + 2\times e^2$

$= 4e^2.$

Now, $h = \frac{b-a}{n}.$

$\therefore \frac{(2-1)}{12} \cdot \left(\frac{2-1}{n}\right)^2 \max f''(x) \leq \frac{1}{3}\times 10^{-6}.$

$$\frac{1}{\cancel{12}_{\cancel{3}}} \times \frac{1}{n^2} \times \cancel{4}e^2 \leq \frac{1}{\cancel{3}} \times 10^{-6}$$

$$\frac{e^2}{n^2} \leq 10^{-6}$$

$\therefore e^2 \times 10^{6} \leq n^2$

$n^2 \geqslant e^2 (10^3)^2$

$n \geqslant e \cdot 10^3$

$n \geqslant 1000 \cdot e.$

$\therefore$ Ans. $1000 \cdot e.$

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Note :-

① Error in $S-\frac{1}{3}$ rule $= \frac{-(b-a)}{180} \cdot h^4 \cdot \max \left[f^{IV}(x) \right]$

— Error order h^4

② Error in $S-\frac{3}{8}$ Rule $= \frac{-3}{80} \cdot h^5 \cdot \max \left[f^{IV}(x) \right]$

— Error order h^5.

IV Solutions to differential Equation

Let us consider differential Eqn.

$$\frac{dy}{dx} = f(x, y) \quad \text{where} \quad y(x_0) = y_0 \text{ ——— } (*)$$

Eqn (*) can be solved by using :-

1) Eulers method
 - forward Eulers method
 - Backward Eulers method.

2) Runge - kutta method
 - Runge Kutta of 1^{st} order - [Eulers method]
 - Runge Kutta of 2^{nd} order - [modified Eulers method]

Not asked in GATE yet. {
 - Runge kutta of 3^{rd} order
 - Runge kutta of 4^{th} order

* Eulers Method :- (forward)

Eulers Iterative formula for finding solution curve to the eq^n (*) is

$$\boxed{y_{n+1} = y_n + h \cdot f(x_n, y_n)}$$

In particular for $n=0$

$$\boxed{y_1 = y_0 + h \cdot f(x_n, y_n)}$$

Q. find an approximate value of 'y' corresponding to $x = 0.2$ & $\frac{dy}{dx} = x + y$, $y = 1$, ~~when~~ $x = 0$. Using Eulers method where $h = 0.1$

make it $f(x, y)$

	x	y	comment
①	$x_0 = 0$	$y_0 = 1$	Initial value.
②	$x_1 = x_0 + h$ $= 0 + 0.1$ $= 0.1$	$y_1 = ?$ $\therefore y_1 = 0.1$	$y_1 = y_0 + h \cdot f(x_0, y_0)$ $= 1 + 0.1\,(x_0 + y_0)$ $= 1 + 0.1\,(0 + 1)$ $y_1 = 1.1$
③	$x_2 = x_1 + h$ $= 0.1 + 0.1$ $= 0.2$	$y_2 = ?$ $\therefore y_2 = 1.22$	$y_2 = y_1 + h \cdot f(x_1, y_1)$ $= 1.1 + 0.1\,(0.1 + 1.1)$ $= 1.1 + 0.1\,(0.2)$ $= 1.22$

Q. $\frac{dy}{dx} - y = x$. when $y(0) = 0$.
where $h = 0.1$

Compute $y(0.3)$ using Euler's 1st order method.

⟹ i.e Euler's forward method.

→ Ans. <u>0.031</u>

x	y	Comment
$x_0 = 0$	0	$y_1 = y_0 + h\,f(x_0, y_0)$
$x_1 = 0.1$	$y_1 = 0$	$y_1 = 0 + 0.1\,(0+0)$ $y_1 = 0$ $y_2 = y_1 + h\,f(x_1, y_1)$
$x_2 = 0.2$	$y_2 = 0.01$	$y_2 = 0 + 0.1\,(0.1+0)$ $= 0.01$ $y_3 = y_2 + h\,f(x_2, y_2)$
$x_3 = 0.3$	$\boxed{y_3 = 0.031}$	$y_3 = 0.01 + 0.1\,(0.2 + 0.01)$ $= 0.01 + 0.1 \times 0.21$ $= 0.01 + 0.021$ $= 0.031$

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* Eulers Backward Method :-

let, $\frac{dy}{dx} = f(x,y)$ ——— (*)

where, $y(x_0) = y_0$

∴ Eulers Backward iterative formula for solving Eqⁿ (*) i.e $\frac{dy}{dx} = f(x,y)$

$$\boxed{y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})}$$

Here, y_{i+1} is in both L.H.S & R.H.S.

Since, y_{i+1} is defined in function

Therefore, this method is called as Implicit Euler's method.

Q.1 find an appropriate value for $x = 0.2$ using Implicit Eulers method where, $\frac{dy}{dx} = x+y$, $y(0) = 1$ where step size $h = 0.1$

→ Given, $\frac{dy}{dx} = x+y$

$$\therefore y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})$$

$$y_{i+1} = y_i + h \cdot (x_{i+1} + y_{i+1})$$

$$(1-h) \cdot y_{i+1} = y_i + h \cdot x_{i+1}$$

$$y_{i+1} = \frac{y_i + h \cdot x_{i+1}}{(1-h)}$$

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In perticular $x=0$ then

$$\text{i.e.}:- \boxed{y_1 = \frac{y_0 + h\cdot x_1}{(1-h)}}$$

∴ Construct the table for y with respective to values of x

x	y	Comment.
$x_0 = 0$	$y_0 = 1$	Initial Condition
$y_1 = 0.1$	$\boxed{y_1 = 1.222}$	$y_1 = \frac{y_0 + h\cdot x_1}{(1-h)}$ $= \frac{1 + 0.1\times 0.1}{(1-0.1)}$ $y_1 = 1.222$
$x_2 = 0.2$	$\boxed{y_2 = 1.38}$	$y_2 = \frac{y_1 + h\cdot x_2}{(1-h)}$ $= \frac{1.222 + 0.1\times 0.2}{(1-0.1)}$ $= 1.38.$

Note:- Eulers Backward method is more stable than forward method.

The exact solution for differential eqn

$\frac{dy}{dx} = x+y$ with $y(0)=1$ is $y = 2e^x - x - 1$

at $x=1$, $y = 3.44$.

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By observing forward & backward Eulers method you can say that Backward method is converging to required value very quickly

GATE Q 2006 The diff. eqn $\frac{dy}{dx} = 0.25y^2$ is to be solved using backward Eulers method with boundary conditions $y=1$ at $x=0$ and $h=1$ what would be the value of y at $x=1$.

(A) 1.33 (B) 1.67 (C) 2.0 (D) 2.33

→ Here, $\frac{dy}{dx} = 0.25y^2$

$$y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1})$$

$$y_{i+1} - 1\left[0.25\, y_{i+1}^2\right] = y_i$$

$$1 \times 0.25\, y_{i+1}^2 - y_{i+1} + y_i = 0.$$

— Here, value of $h=1$, ∴ No need to construct table

$$0.25\, y_{i+1}^2 - y_{i+1} + y_i = 0.$$

Comparing with $ax^2 + bx + c = 0$

∴ roots of Eqn. are

$$\cancel{x = \frac{-b \pm \sqrt{4}}{}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2 \cdot a}$$

$$\therefore y_{i+1} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(0.25)(y_i)}}{2 \times 0.25}$$

$$y_{i+1} = \frac{1 \pm \sqrt{1 - y_i}}{0.5}$$

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put $i=0$,

$$y_1 = \frac{1 \pm \sqrt{1-y_0}}{0.5} \quad \text{——} \quad y_0 = 1.$$

i.e. $\underline{y(0)=1}$ — given

$$\therefore y_1 = \frac{1 \pm \sqrt{1-1}}{0.5}$$

$$y_1 = \frac{1}{0.5}$$

$$\boxed{y_1 = 2}$$

Runge Kutta Method :-

GATE 1996 2nd order R-K → Given diff. eq^n $\frac{dy}{dx} = x - y$ with $y(0)=0$ then value of $y(0.1)$ using 2^{nd} order Runge Kutta method, with step size $h=0.1$

Runge kutta method of 2^{nd} order Iterative formula for finding sol^n curve to eq^n i)

$$\boxed{y_1 = y_0 + \frac{1}{2}(k_1+k_2)}$$

where $\boxed{k_1 = h\,f(x_0,y_0)}$

$k_1 = 0.1 \times (x_0 - y_0)$

$= 0.1 \times (0-0)$

$\boxed{k_1 = 0}$

$\boxed{k = h\,f(x_0+h, y_0+k_1)}$

$k_2 = 0.1[(x_0+h) - (y_0+k_1)]$

$= 0.1[(0+0.1) - (0+0)]$

$\boxed{k_2 = 0.01}$

$$\therefore y_1 = y(x_0+h) = y(0+0.1) = y(0.1)$$

$$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$$

$$= 0 + \frac{1}{2}(0+0.01)$$

$$= \frac{0.01}{2}$$

$$\boxed{y_1 = 0.005}$$

Q.2. 4th Order R-K → Apply Runge-kutta method of 4^{th} order where $\frac{dy}{dx} = x+y$, $y=1$ when $x=0$, $h=0.2$

Compute $y(0.2)$

Runge kutta method of 4^{th} order formula

$$\boxed{y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

where

$k_1 = h \cdot f(x_0, y_0)$	$k_2 = h \cdot f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$	$k_3 = h\,f\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$
$= 0.2(x_0+y_0)$ $= 0.2(0+1)$ $k_1 = 0.2$	$= 0.2\left[0+\frac{0.2}{2}+1+\frac{0.2}{2}\right]$ $k_2 = 0.24$	$= 0.2\left[0+\frac{0.2}{2}+1+\frac{0.24}{2}\right]$ $= 0.2[0.1+1+0.12]$ $k_3 = 0.244$

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&

$$\boxed{k_4 = h\cdot f(x_0+h,\ y_0+k_3)}$$

$$= 0.2\left[x_0+h + y_0+k_3\right]$$

$$= 0.2\left[0+0.2+1+0.2+4\right]$$

$$k_4 = 0.2888$$

$$\therefore y_1 = y_0 + \frac{1}{6}\left[k_1+2k_2+2k_3+k_4\right]$$

$$= 1+\frac{1}{6}\left(0.2+2\times 0.24+2\times 0.244+0.288\right)$$

$$\boxed{y_1 = 1.2428}$$

Q.3. 3rd order R-K

Apply Runge kutta method of 3^{rd} order with $\frac{dy}{dx} = x+y$, when $y=1$, $x=0$ & $h=0.2$ then compute $y(0.2)$

→ Runge Kutta method of 3^{rd} order iterative formula for finding Solution Curve to the Eq^n is.

$$\boxed{y_1 = y_0 + \frac{1}{6}\left(k_1+4k_2+k_3\right)}$$

where,

$$\boxed{k_1 = h\cdot f(x_0,y_0)}$$

$$= 0.2\,(x_0+y_0)$$

$$= 0.2\,(0+1)$$

$$k_1 = 0.2$$

$$\boxed{k_2 = h\cdot f\left(x_0+\frac{h}{2},\ y_0+\frac{k_1}{2}\right)}$$

$$= 0.2\left[0+\frac{0.2}{2}+1+\frac{0.2}{2}\right]$$

$$k_2 = 0.24$$

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$$\boxed{k_3 = h\cdot f(x_0+h,\ y_0+k')}$$

where, ~~$k'=0$~~

$$\boxed{k' = h\cdot f(x_0+h,\ y_0+k_1)}$$

$\therefore k' = 0.2\,[0+0.2+1+0.2]$

$= 0.2\,(1.4)$

$k' = 0.28$

$k_3 = 0.2\,[0+0.2+1+0.28]$

$k_3 = 0.296$

$y_1 = y_0 + \frac{1}{6}[k_1 + 4k_2 + k_3]$

$\therefore y_0 = 1 + \frac{1}{6}[0.2 + 4\times 0.24 + 0.296]$

$$\boxed{y_0 = 1.2426}$$

Q.4. GATE 2007 Euler's method

The diff. eq^n $\frac{dx}{dt} = \frac{1-x}{\tau}$ is evaluated using Eulers method with step size $h=\Delta T$, where $\Delta T > 0$ what is the maximum value of ΔT, To Ensure Stability in sol^n

(A) 1 (B) $\frac{\tau}{2}$ (C) τ (D) 2τ ✓

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→ Stable :-

An iterative method is said to be stable if the round off error is remains bounded as $n \to \infty$, where n is no. of Iterations.

The Eulers method formula

$$\boxed{y_{i+1} = y_i + h \cdot f(x_i, y_i)}$$ can be written as

$$\boxed{y_{i+1} = E \cdot y_i + K} \quad \text{———} (*)$$

where,

K = terms which are involved in x or Consts.

Eq^n (*) is said to be stable, if $|E| < 1$

i.e $-1 \leq |E| < 1$

$$\frac{dx}{dt} = \frac{1-x}{\tau}$$

Similarly $\frac{dy}{dx} = \frac{1-y}{\tau}$

$$\therefore \frac{dy}{dx} = \frac{1-y}{\tau} = f(x, y)$$

$$\therefore y_{i+1} = y_i + h \cdot f(x_i, y_i)$$
$$= y_i + h \cdot \left(\frac{1-y_i}{\tau}\right)$$

$$y_{i+1} = \underbrace{\left(1 - \frac{h}{\tau}\right)}_{E} \cdot y_i + \underbrace{\frac{h}{\tau}}_{K} \quad \text{———} (I)$$

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Eq^n (I) is stable if $\left|1-\frac{h}{\tau}\right|<1$

$\therefore \left|1-\frac{h}{\tau}\right|<1$

$\left|1-\frac{\Delta T}{\tau}\right|<1$

i.e. $-1<\left|1-\frac{\Delta T}{\tau}\right|<1$

Substract 1 throughout to reduce 1 or cancell 1 from middle term

$\therefore -1-1<\left|1-1-\frac{\Delta T}{\tau}\right|<1-1$

$\therefore -2<\left|-\frac{\Delta T}{\tau}\right|<0$

$\therefore -2\tau<|-\Delta T|<0$

$\therefore 2\tau>\Delta T>0$ —— Removing signs and changing directions of Equality signs.

i.e. $0<\Delta T<2\tau$

$\therefore \Delta \tau<2\tau$

Ans. 2τ

Q.1. The min. no. of equal length subintervals needed to approximate $\int_0^2 e^{2x}dx$ to an accuracy of atleast $\frac{8}{45}\times10^{-8}$ using Simpson's Rule

Simpson Rule

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(A) 200 e (B) 200 (C) 2000e (D) 2000

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$\rightarrow$ $f(x) = e^{2x}$ —— $[a,b] = [0,2]$

Accuracy atleast means

$$\text{accuracy} \geq \frac{8}{45} \times 10^{-8}$$

Here, if Accuracy ↑ then Error ↓

$$\therefore \text{Error} \leq \frac{8}{45} \times 10^{-8}$$

In Numerical method error is considered as ~~Simpson's~~ Truncation method

$$\left|\left[\text{Truncation Error}\right]_{\text{Simpson rule}}\right| \leq \frac{8}{45} \times 10^{-8}$$

$$\left|\frac{b-a}{180} \times h^4 \times \text{Max}\left(f^{IV}(x)\right)\right| \leq \frac{8}{45} \times 10^{-8}$$

But $h = \dfrac{b-a}{n}$

$$\therefore \left|\frac{2-0}{180} \times \left(\frac{2-0}{n}\right)^4 \times \text{Max}.\, f^{IV}(x)\right| \leq \frac{8}{45} \times 10^{-8}$$

Now, $f(x) = e^{2x}$

$\therefore f'(x) = 2 \cdot e^{2x}$

$f''(x) = 4 \cdot e^{2x}$

$f'''(x) = 8 \cdot e^{2x}$

$f^{IV}(x) = 16 \cdot e^{2x} = 16 \times e^{2 \times 2} = 16 \cdot e^4$ — last limit ~~for~~ x is 2

$$\therefore \left|\frac{\cancel{2}-0}{\cancel{180}_{4}} \times \left(\frac{2-0}{n}\right)^{\cancel{4}} \times \cancel{16} \cdot e^4\right| \leq \frac{8}{\cancel{45}} \times 10^{-8}$$

$$\therefore \left| \left(\frac{2}{n}\right)^4 \cdot e^4 \right| \leq \frac{8}{1} \times 10^{-8}$$

$$\left| \frac{16}{n^4} \cdot e^4 \right| \leq 8 \times 10^{-8}$$

$$\left| \frac{16 \cdot e^4}{1 \times 10^{-8}} \right| \leq n^4$$

$$\left| 16 \times e^4 \times 10^{+8} \right| \leq n^4$$

$$\therefore n^4 \geq \left| 16 \times e^4 \times 10^{+8} \right|$$

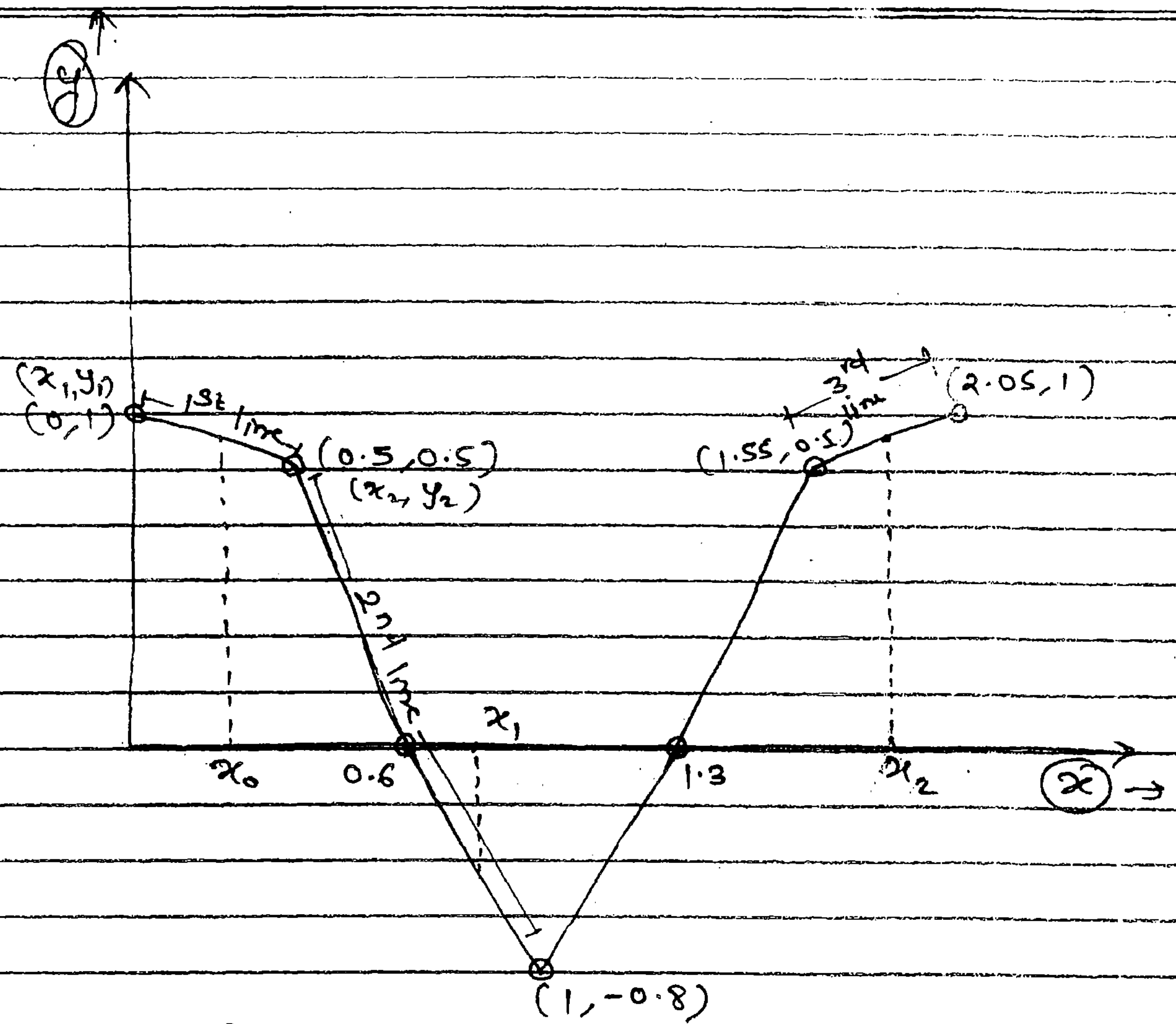
$$\therefore n^4 \geq \left| 2^4 \times e^4 \times (10^2)^4 \right|$$

$$\therefore n \geq \left| 2 \times e^4 \times 10^2 \right|$$

$$n \geq 200\, e^4 \quad \text{———— Ans.}$$

Q. 2.
GATE If we use Newton raphson method to find roots $f(x) = 0$, Using x_0, x_1 & x_2 respectively As initial guess values & then the roots Obtained would be

(A) 1.3, 0.6, 0.6
(B) 0.6, 0.6, 1.3
(C) 1.3, 1.3, 0.6
(D) 1.3, 0.6, 1.3



→ 1st line for x_0

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad (0,1) \longrightarrow (0.5, 0.5)$$

$(0.1, 1)$ — $(0.5, 0.5)$ ← original

$$m = \frac{0.5 - 1}{0.5 - 0}$$

$$m = -1$$

Eqn of 1st line $\Rightarrow (y - y_1) = m(x - x_1)$

$$(y - 1) = -1\,(x - 0)$$

$$\underline{\underline{x + y = 1}} \Rightarrow y = -x + 1$$

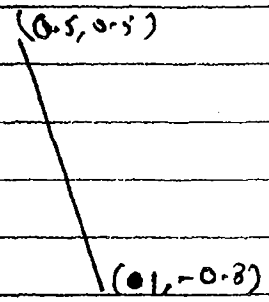
$$\Rightarrow y = mx + c$$

$\Rightarrow c \to 1$

$\therefore$ converges to 1.3

On x axis, $y=0$,
So, $x=1$ which is near to 1.3 than 0.6 on x axis $\therefore$ x_0 is Converging to 1.3.

– 2^{nd} line for x_1 $(0.5, 0.5)$ $(1, -0.8)$



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-0.8 - 0.5}{1 - 0.5}$$

$$m = -1.8$$

Eq^n of 2^{nd} line $\Rightarrow$ $(y - y_1) = m(x - x_1)$

$$(y - 0.5) = -1.8(x - 0.5)$$

$$y - 0.5 = -1.8x + 0.9$$

$$y + 1.8x = 0.9 + 0.5$$

$$\underline{\underline{y + 1.8x = 1.3}}$$

$$\Rightarrow y = -1.8x + 1.3$$

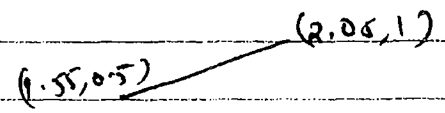
$$\Rightarrow y = mx + c$$

$$\Rightarrow c \rightarrow 1.3.$$

$\therefore$ Converges to 1.3.

– 3^{rd} line for x_2 $(1.55, 0.5)$ $(2.05, 1)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0.5}{2.05 - 1.55}$$



$$m = 0$$

Eq^n

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Eq of 3rd line

$(y - y_1) = m(x - x_1)$

$\cancel{y = 1.55}$)

$(y - 0.5) = 0(x - 1.55)$

$\therefore \quad y - 0.5 = 0$

$y = 0.5$

$\Rightarrow \therefore y = mx + c$

$\therefore c \Rightarrow 0.5$

Converges to $\underline{0.6}$